

# Quantum Chaos in Multi–Matrix Models

E.Vinteler

International School for Advanced Studies (SISSA/ISAS)

Via Beirut 2, 34014 Trieste, Italy

## Abstract

We propose a possible resolution for the problem of why the semicircular law is not observed, whilst the random matrix hypothesis describes well the fluctuation of energy spectra. We show in the random 2-matrix model that the interactions between the quantum subsystems alter the semicircular law of level density. We consider also other types of interactions in the chain- and star-multimatrix models. The connection with the Calogero-Sutherland models is briefly discussed.

# 1 Introduction

In heavy nuclei, the complicated many-body interactions lead to statistical theories which explain only the average properties. One of these theories is the random matrix hypothesis [1][2]. It supposes that the nuclear hamiltonian in a arbitrary basis of functions is a  $N \times N$  matrix with  $N$  large and elements distributed at random. The joint probability function of the eigenvalues  $\lambda_1, \dots, \lambda_N$  of this matrix model is given by:

$$P(\lambda_1, \dots, \lambda_N) = \exp\left(-\sum_{i=1}^N \lambda_i^2\right) \prod_{i < j} (\lambda_i - \lambda_j)^\beta \quad (1.1)$$

where  $\beta = 1, 2, 4$  for orthogonal, hermitean and, respectively unitary ensembles. Integrating over eigenvalues  $\lambda_{k+1} \dots \lambda_N$  we get the joint distribution function for few levels:

$$P(\lambda_1, \dots, \lambda_k) = \int d\lambda_{k+1} \dots d\lambda_N P(\lambda_1, \dots, \lambda_N) \quad (1.2)$$

All these joint distribution functions can be expressed in terms of the Dyson correlation function  $K(\lambda_1, \lambda_2)$ :

$$P(\lambda_1, \dots, \lambda_k) = \sum_{\sigma} (-1)^\sigma K(\lambda_1, \lambda_{\sigma_1}) \dots K(\lambda_k, \lambda_{\sigma_k})$$

where  $\sigma$  is the permutation of  $k$  levels. In the special case  $k = 1$  the Dyson correlation function coincides with level density  $K(\lambda, \lambda) = P(\lambda)$ .

The density of levels for the 1-matrix model satisfies the semicircular law:

$$P(\lambda) = \sqrt{\beta N/2 - \lambda^2}$$

and the Dyson correlation function behaves as ( $\sigma \ll \lambda$ ):

$$K\left(\lambda - \frac{1}{2}\sigma, \lambda + \frac{1}{2}\sigma\right) \simeq \frac{\sin(\pi\sigma P(\lambda))}{\pi\sigma(\beta N/2)}$$

The Dyson correlation function describes well the fluctuations of quantum systems, but the semicircular law is not observed in the experimental data for the density of levels. A possible resolution of problem is to consider instead one random matrix few random matrices in interaction. As we will see, even a small interaction gives a qualitatively new behaviour for the level density.

As an interesting generalization of the random matrix hypothesis is to consider  $q$  matrices describing  $q$  nuclear systems in interaction. The total action of such system is:

$$S_1 = \sum_{\alpha=1}^q \sum_{i=1}^N (t_\alpha (\lambda_i^{(\alpha)})^2 + u_\alpha \lambda_i^{(\alpha)}) + \sum_{\alpha=1}^{q-1} \sum_{i=1}^N c_\alpha \lambda_i^{(\alpha)} \lambda_i^{(\alpha+1)} \quad (1.3)$$

This system describes a chain of matrices with neighbour interaction. We can add a term describing the two-body interaction of constituent nuclear subsystems:

$$\sum_{|\alpha-\beta| \neq 1} \sum_{i=1}^N c_{\alpha,\beta} \lambda_i^{(\alpha)} \lambda_i^{(\beta)}$$

We have different sets of energy levels  $\lambda_1^{(\alpha)}, \dots, \lambda_N^{(\alpha)}, \alpha = 1 \dots q$  with distribution probability:

$$P(\lambda_1^{(1)}, \dots, \lambda_N^{(1)} \dots \lambda_1^{(q)}, \dots, \lambda_N^{(q)}) = \exp(S) \prod_{i < j} (\lambda_i^{(1)} - \lambda_j^{(1)}) (\lambda_i^{(q)} - \lambda_j^{(q)}) \quad (1.4)$$

We have level repulsion only for the first and last energy level set. Hence for this model the intermediate energy level sets are "classical" and interact with "quantum" first and last energy level sets. Integrating over all intermediate matrices we remain with a two-matrix model.

Kharchev and others have considered the so-called conformal matrix models that contain additional repulsion terms also for intermediate matrices [3].

Another special random matrix model is the star-matrix model having the action:

$$S_2 = \sum_{i=1}^N (t_0(\lambda_i^{(0)})^2 + u_0 \lambda_i^{(0)}) + \sum_{\alpha=1}^q \sum_{i=1}^N (t_\alpha(\lambda_i^{(\alpha)})^2 + u_\alpha \lambda_i^{(\alpha)}) + \sum_{\alpha=1}^q \sum_{i=1}^N c_\alpha \lambda_i^{(\alpha)} \lambda_i^{(0)} \quad (1.5)$$

The joint distribution of this model reduces again to that of 2-matrix model.

## 2 Quantum Chaos in two-matrix model

We introduce the distribution probability:

$$P(\lambda_1 \dots \lambda_N, \mu_1 \dots \mu_N) = \exp \sum_{i=1}^N (V_1(\lambda_i) + V_2(\mu_i) + c_i \lambda_i \mu_i) \prod_{i < j} (\lambda_i - \lambda_j) (\mu_i - \mu_j) \quad (2.1)$$

with  $V_\alpha(\tau) = t_\alpha \tau^2 + u_\alpha \tau, \alpha = 1, 2$  and joint distribution function :

$$P(\lambda_1 \dots \lambda_i, \mu_1 \dots \mu_j) = \int d\lambda_{i+1} \dots d\lambda_N, d\mu_{j+1} \dots d\mu_N P(\lambda_1 \dots \lambda_N, \mu_1 \dots \mu_N) \quad (2.2)$$

We show that the level densities  $P(\lambda), P(\mu)$  and the joint probability distributions  $P(\lambda_1, \lambda_2), P(\mu_1, \mu_2)$  are exactly like those of the hermitean 1-matrix model with distribution probability (1.1):

$$\begin{aligned} P(\lambda) &= P_{Herm}(\lambda'), P(\mu) = P_{Herm}(\mu') \\ P(\lambda_1, \lambda_2) &\sim P_{Herm}(\lambda'_1, \lambda'_2), P(\mu_1, \mu_2) \sim P_{Herm}(\mu'_1, \mu'_2) \end{aligned} \quad (2.3)$$

The new joint probability distributions  $P(\lambda, \mu)$  behaves in a different way because we have not energy repulsion between levels of different sets.

If we set from beginning the coupling  $c = 0$  we get two independent orthogonal 1-matrix models and we have:

$$P(\lambda, \mu) = P_{Orth}(\lambda') P_{Orth}(\mu')$$

For  $c \neq 0$ ,  $P(\lambda, \mu)$  behaves like the 1-matrix Dyson correlation function:

$$P(\lambda, \mu) \sim K(\lambda, \mu)$$

When  $c \rightarrow 0, P(\lambda, \mu)$  does not split in two orthogonal 1-matrix models.

Here  $\lambda', \mu'$  are related with the coefficients of the  $Q$ -matrices.

$$\lambda' = \frac{\lambda - a_0}{\sqrt{2a_1}}, \mu' = \frac{\mu - b_0}{\sqrt{2b_1}} \quad (2.4)$$

with:

$$\begin{aligned} a_0 &= \frac{c_1 u_2 - 2t_2 u_1}{4t_1 t_2 - c_1^2}, b_0 = \frac{c_1 u_1 - 2t_1 u_2}{4t_1 t_2 - c_1^2}, \\ a_1 &= -\frac{2t_2}{4t_1 t_2 - c_1^2}, b_1 = -\frac{2t_1}{4t_1 t_2 - c_1^2} \end{aligned} \quad (2.5)$$

In the rest of the section we demonstrate the above relations. We introduce the  $Q$ -matrices, which in the two-matrix case have the form:

$$Q_1 = I_+ + a_0 I_0 + a_1 \epsilon_-, \bar{Q}_2 = I_+ + b_0 I_0 + b_1 \epsilon_-, \quad (2.6)$$

with

$$I_+ = \sum_{n=0}^N E_{n,n+1}, I_0 = \sum_{n=0}^N E_{n,n}, \epsilon_- = \sum_{n=0}^N n E_{n,n-1}$$

The  $Q$  matrices are defined as:

$$\int d\lambda d\mu \xi_n(\lambda) \lambda_\alpha e^{V_1 + V_2 + c\lambda\mu} \eta_m(\mu) = Q_{\alpha,nm} h_m, \alpha = 1, 2$$

where  $h_n = h_0 R^n$  and  $R = c/(4t_2 t_1 - c^2)$ .

$\xi, \eta$  are orthogonal polynomials

$$\xi_n(\lambda) = \lambda_1^n + \dots, \eta_n(\mu) = \mu^n + \dots$$

satisfying the orthogonality condition:

$$\int d\lambda d\mu \xi_n(\lambda) e^{V_1 + V_2 + c\lambda\mu} \eta_m(\mu) = h_n \delta_{nm} \quad (2.7)$$

From the definition of  $Q$ -matrices ( $Q_{1,mn} \xi_m = \lambda_n \xi_n, \bar{Q}_{2,mn} \eta_m = \mu_n \eta_n$ ) we have the following recursion relations of the orthogonal polynomials:

$$\begin{aligned} \lambda \xi_n(\lambda) &= \xi_{n+1}(\lambda) + a_0 \xi_n(\lambda) + a_1 \xi_{n-1}(\lambda) \\ \mu \eta_n(\mu) &= \eta_{n+1}(\mu) + b_0 \eta_n(\mu) + b_1 \eta_{n-1}(\mu) \end{aligned} \quad (2.8)$$

Solving these recursion relations it follows that  $\xi, \eta$  are Hermite functions:

$$\xi_n(\lambda) = \alpha_n H_n(\lambda'), \eta_m(\mu) = \beta_m H_m(\mu')$$

To get the proportionality coefficients  $\alpha_n, \beta_m$  we use the orthogonality relation and the Gauss transform:

$$(2\pi u)^{-1/2} \int dy e^{-(x-y)^2/(2u)} H_n(y) = (1-2u)^{n/2} H_n((1-2u)^{-1/2} x) \quad (2.9)$$

Writing the action as:

$$\begin{aligned} S &= V_1(\lambda) + V_2(\mu) + c\lambda\mu = t_1\lambda^2 + u_1\lambda + t_2\mu^2 + u_2\mu + c\lambda\mu = \\ &= S_0 + t_2 \left( \mu + \frac{u_2 + c\lambda}{2t_2} \right)^2 - \left( \frac{\lambda - a_0}{\sqrt{2a_1}} \right)^2 \end{aligned} \quad (2.10)$$

with:

$$S_0 = -\frac{t_1 u_2^2 + t_2 u_1^2 - c u_1 u_2}{4t_1 t_2 - c^2} \quad (2.11)$$

we have:

$$\begin{aligned} \int d\lambda d\mu \xi_n(\lambda) e^{V_1 + V_2 + c\lambda\mu} \eta_m(\mu) &= \alpha_n \beta_m \delta_{nm} e^{S_0} \frac{2\pi}{\sqrt{4t_1 t_2 - c^2}} \left( \frac{c}{\sqrt{4t_1 t_2 - c^2}} \right)^n 2^n n! = \\ &= h_0 \delta_{nm} R^n = h_0 \left( \frac{c}{4t_1 t_2 - c^2} \right)^n \end{aligned} \quad (2.12)$$

In conclusion:

$$\begin{aligned} \xi_n(\lambda) &= (2\pi n!)^{-1/2} 2^{-n/2} (\sqrt{2a_1})^n H_n(\lambda'), \\ \eta_m(\mu) &= (2\pi m!)^{-1/2} 2^{-m/2} (\sqrt{2b_1})^m H_m(\mu') \end{aligned} \quad (2.13)$$

and:

$$h_0 = (4t_1 t_2 - c^2)^{-1/2} \exp(S_0)$$

We can now calculate the joint probability distribution  $P(\lambda, \mu)$ . Because we can write the two Vandermonde determinants in terms of orthogonal polynomials  $\xi_n, \eta_m$

$$\Delta(\lambda) \Delta(\mu) = \sum_n \xi_n(\lambda_1) \Xi_n(\lambda_2 \dots \lambda_N) \sum_m \eta_m(\mu_1) \Theta_m(\mu_2 \dots \mu_N)$$

and the algebraic complements satisfy:

$$\int \prod_{i=2}^N (d\lambda_i d\mu_i) \Xi_n(\lambda_2 \dots \lambda_N) \Theta_m(\mu_2 \dots \mu_N) = (N-1)! \delta_{nm}$$

we get for joint probability distribution:

$$P(\lambda, \mu) = \frac{1}{N} e^S \sum_{n=0}^{N-1} h_n^{-1} \xi_n(\lambda) \eta_n(\mu) \quad (2.14)$$

It is easy to derive the expression for symmetric joint distribution of pairs of eigenvalues in terms of  $P(\lambda, \mu)$ :

$$P(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k) = \sum_{\sigma} (-1)^{\sigma} P(\lambda_1, \mu_{\sigma_1}) \dots P(\lambda_k, \mu_{\sigma_k})$$

Integrating in  $\lambda_{j+1} \dots \lambda_k$  we obtain the asymmetric joint distribution of eigenvalues:

$$P(\lambda_1, \dots, \lambda_j, \mu_1, \dots, \mu_k) = \sum_{\sigma} (-1)^{\sigma} P(\lambda_1, \mu_{\sigma_1}) \dots P(\lambda_j, \mu_{\sigma_j}) P(\mu_{\sigma_{j+1}}) P(\mu_{\sigma_k})$$

In the limit of large  $N$  we have the usual behaviour of semi-circular law:

$$P(\lambda) = \sqrt{2N - \lambda'^2}, P(\mu) = \sqrt{2N - \mu'^2}$$

To calculate the joint distribution of two eigenvalues  $P(\lambda, \mu)$  in the large  $N$  limit we associate it with the quantum mechanical system :

$$\left[\frac{1}{2}(p_\lambda^2 + p_\mu^2) + V_1(\lambda) + V_2(\mu) + c\lambda\mu\right]\phi_n(\lambda)\psi_m(\mu) = E_{nm}\phi_n(\lambda)\psi_m(\mu)$$

where  $p_\lambda = i\partial/\partial\lambda, p_\mu = i\partial/\partial\mu$  are the usual momenta operators and

$$\begin{aligned}\phi_n(\lambda) &= \exp(-\lambda'^2/2)\eta_n(\lambda) \\ \psi_m(\mu) &= \exp(-\mu'^2/2)\xi_m(\mu)\end{aligned}\tag{2.15}$$

For  $c = 0$  we get two decoupled quantum systems:

$$\begin{aligned}(p_\lambda^2 + \lambda'^2)\phi_n(\lambda) &= 2E_{1,n}\phi_n(\lambda) \\ (p_\mu^2 + \mu'^2)\psi_m(\mu) &= 2E_{2,m}\psi_m(\mu)\end{aligned}\tag{2.16}$$

where  $E_{nm} = E_{1,n} + E_{2,m}$ .

In the large  $N$  limit  $E_{nm}$  behaves like  $\sim N$  and because we are searching for symmetric solutions we have  $E_{1,n} = E_{2,m} \sim N/2$ . The joint distribution of two eigenvalues  $P(\lambda, \mu)$  will be:

$$P(\lambda, \mu) = \sqrt{2E_{1,n} - \lambda'^2}\sqrt{2E_{2,n} - \mu'^2}\tag{2.17}$$

or

$$P(\lambda, \mu) = \sqrt{N - \lambda'^2}\sqrt{N - \mu'^2}\tag{2.18}$$

We can see that for  $c = 0$ ,  $P(\lambda, \mu)$  is the product of density energy levels for orthogonal ansembles. If we integrate the last matrix, we get the 1-matrix model. In our case this is equivalent with the condition  $2E_{2,m} = p_\mu^2 + V_2(\mu) = 0$  in (2.16) or in other words the second system has no contribution in the joint distribution of two eigenvalues. The equation (2.17) is replaced by:

$$P(\lambda) = \sqrt{2N - \lambda'^2}$$

For  $c \neq 0$ , after summing relation (2.14) and using the asymptotic formula ( $n$  large) for the Hermite polynomial (near origin):

$$H_n = e^{x^2} \frac{\Gamma(n+1)}{\Gamma(n/2+1)} \cos(\sqrt{2n+1} - \frac{n\pi}{2}) + O(1/\sqrt{n})$$

we obtain (up the exponent  $S + (\lambda'^2 + \mu'^2)/2$ ):

$$P(\lambda, \mu) \sim \frac{\sin \sqrt{2N}(\lambda' - \mu')}{\pi N(\lambda' - \mu')}, \quad \lambda, \mu \text{ near } 0\tag{2.19}$$

We also get for arbitrary  $\lambda, \mu, \lambda \ll \mu$ :

$$P(\lambda, \mu) \sim \frac{\sin \sqrt{2N - (\alpha\lambda)^2} \epsilon(\alpha\lambda)}{\pi N \epsilon(\alpha\lambda)} \quad (2.20)$$

where:

$$\begin{aligned} \epsilon &= \frac{1}{\sqrt{2a_1}} - \frac{1}{\sqrt{2b_1}} \\ \alpha &= \frac{1}{2} \left( \frac{1}{\sqrt{2a_1}} - \frac{1}{\sqrt{2b_1}} \right) \end{aligned} \quad (2.21)$$

For the asymmetric potential  $t_1 = 1/(a + \tau)^2, t_2 = 1/(a - \tau)^2, (\tau \ll a)$  and a small interaction  $c \approx 0$ , we have  $\epsilon \sim \tau/a^2, \alpha \sim 1/(2a)$  and  $\epsilon \ll \alpha$ . When  $\tau \rightarrow 0$  (symmetric potential)  $P(\lambda, \mu \sim \lambda)$  tends to the level density of hermitean 1-matrix model  $P_{Herm}(\lambda)$ . The interaction (even a small one) of asymmetric energy levels changes dramatically the level density  $P(\lambda, \lambda)$  of the system.

If for  $\tau \rightarrow 0$  we get the usual semicircular law, a small asymmetry creates some peaks in the level density  $P(\lambda, \lambda)$  (see figure 1). The observed behaviour is the quantum analog for chaotical behaviour of two interacting classical oscillators.

### 3 $q$ -matrix model

As a random  $q$ -multimatrix model we choose the one with partition function:

$$Z = \int \prod_{\alpha=1}^q \lambda_\alpha \Delta(\lambda_1) \Delta(\lambda_q) \exp \left( \sum_{\alpha=1}^q t_\alpha \lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha \lambda_\alpha \lambda_{\alpha+1} \right) \quad (3.1)$$

We show that the joint probability is :

$$P(\lambda_\alpha, \lambda_\beta) = P_{Herm}(\lambda'_\alpha, \lambda'_\beta), 1 \leq \alpha \leq \beta \leq q \quad (3.2)$$

where:

$$\lambda'_\alpha = \lambda_\alpha / \sqrt{2a_\alpha},$$

The parameters  $a_\alpha$  are the coefficients of the  $Q$ -matrices.

The  $Q(\alpha)$  have only three non-vanishing diagonal lines, the main diagonal and the two adjacent lines.

$$Q(\alpha) = b_\alpha I_+ + a_\alpha \epsilon_- \quad (3.3)$$

where in the particular cases we know that  $b_1 = 1$  and  $a_q = R$ . We can write the parameters in terms of the determinants of two matrices (we use the results of the paper [4]):

$$\begin{aligned} b_\alpha &= (-1)^\alpha (c_1 c_2 \dots c_{\alpha-1})^{-1} \det X_{\alpha-1} \\ R &= (-1)^q c_1 c_2 \dots c_{q-1} \left( \det X_q \right)^{-1} \\ a_\alpha &= (-1)^\alpha c_1 c_2 \dots c_{\alpha-1} \frac{\det Y_{\alpha+1}}{\det X_q} \end{aligned} \quad (3.4)$$

The matrices  $X_\alpha$  and  $Y_\alpha$ , are defined as follows

$$X_\alpha = \begin{pmatrix} 2t_1 & c_1 & 0 & \dots & 0 & 0 \\ c_1 & 2t_2 & c_2 & \dots & 0 & 0 \\ 0 & c_2 & 2t_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2t_{\alpha-1} & c_{k-1} \\ 0 & 0 & 0 & \dots & c_{k-1} & 2t_\alpha \end{pmatrix} \quad (3.5)$$

and

$$Y_\alpha = \begin{pmatrix} 2t_\alpha & c_k & 0 & \dots & 0 & 0 \\ c_k & 2t_{\alpha+1} & c_{k+1} & \dots & 0 & 0 \\ 0 & c_{k+1} & 2t_{\alpha+2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2t_{q-1} & c_{q-1} \\ 0 & 0 & 0 & \dots & c_{q-1} & 2t_q \end{pmatrix} \quad (3.6)$$

Of course  $Y_1 \equiv X_q$ .

As we made before for the 2-matrix model we introduce the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \quad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \dots d\lambda_q \xi_n(\lambda_1) e^S \eta_m(\lambda_q) = h_n \delta_{nm} \quad (3.7)$$

where

$$S = \sum_{\alpha=1}^q t_\alpha \lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha \lambda_\alpha \lambda_{\alpha+1}.$$

We introduce also the basic intermediate functions:

$$\xi_n^{(\alpha)}(\lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \xi_n(\lambda_1) e^{S_\alpha}. \quad (3.8)$$

and

$$\eta_n^{(\alpha)}(\lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^q d\lambda_\beta e^{S'_\alpha} \eta_m(\lambda_q). \quad (3.9)$$

where we denote

$$S_\alpha = \sum_{\beta=1}^{\alpha-1} t_\beta \lambda_\beta^2 + \sum_{\beta=1}^{\alpha-1} c_\beta \lambda_\beta \lambda_{\beta+1}.$$

$$S'_\alpha = \sum_{\beta=\alpha+1}^{q-1} t_\beta \lambda_\beta^2 + \sum_{\beta=\alpha}^{q-1} c_\beta \lambda_\beta \lambda_{\beta+1}.$$



Obviously we have

$$\xi_n^{(1)}(\lambda_1) = \xi_n(\lambda_1), \quad \eta_n^{(q)}(\lambda_q) = \eta_n(\lambda_q).$$

In the general case when we have arbitrary potentials one sees immediately that  $\xi^{(\alpha)}$  and  $\eta^{(\alpha)}$  are not polynomials anymore. In our case of gaussian potentials these intermediate functions are again Hermite functions, but with different arguments. However they still satisfy an orthogonality relation

$$\int \prod_{\gamma=\alpha}^{\beta} d\lambda_{\gamma} \xi_n^{(\alpha)}(\lambda_{\alpha}) e^{S-S_{\alpha}-S'_{\alpha}}(\lambda_{\alpha}) \eta_m^{(\beta)}(\lambda_{\beta}) = \delta_{nm} h_n, \quad 1 \leq \alpha \leq \beta \leq q. \quad (3.10)$$

These basic intermediate functions permit to write the intermediate  $Q$  matrices as:

$$\int d\lambda_{\alpha} \xi_n^{(\alpha)}(\lambda_{\alpha}) e^{V_{\alpha}(\lambda_{\alpha})} \lambda_{\alpha} \eta_m^{(\alpha)}(\lambda_{\alpha}) = Q_{\alpha,nm} h_m = \bar{Q}_{\alpha,nm} h_n, \quad 1 \leq \alpha \leq q. \quad (3.11)$$

The equations satisfied by basic intermediate functions are:

$$\lambda_{\alpha} \xi^{(\alpha)} = Q_{\alpha} \xi^{(\alpha)}, \quad 1 \leq \alpha \leq q. \quad (3.12)$$

$$\lambda_{\alpha} \eta^{(\alpha)} = \bar{Q}_{\alpha} \eta^{(\alpha)}, \quad 1 \leq \alpha \leq q. \quad (3.13)$$

These equations together with the explicit form of  $Q$ -matrices permits to find the basic intermediate functions  $\xi_n^{(\alpha)}$ ,  $\eta_m^{(\alpha)}$ :

$$\begin{aligned} \lambda_{\alpha} \xi_n^{(\alpha)}(\lambda_{\alpha}) &= b_{\alpha} \xi_{n+1}(\lambda_{\alpha}) + a_{\alpha} \xi_{n-1}(\lambda_{\alpha}) \\ \lambda_{\alpha} \eta_m^{(\alpha)}(\lambda_{\alpha}) &= (a_{\alpha}/R) \eta_{m+1}(\lambda_{\alpha}) + b_{\alpha} R \eta_{m-1}(\lambda_{\alpha}) \end{aligned} \quad (3.14)$$

Solving these recursion relations it follows that  $\xi^{(\alpha)}, \eta^{(\alpha)}$  are Hermite functions for gaussian potentials:

$$\begin{aligned} \xi_n^{(\alpha)}(\lambda_{\alpha}) &= (2\pi n!)^{-1/2} 2^{-n/2} (\sqrt{2a_{\alpha}})^n H_n(\lambda'_{\alpha}), \\ \eta_m^{(\alpha)}(\lambda_{\alpha}) &= (2\pi m!)^{-1/2} 2^{-m/2} \left(\frac{R}{\sqrt{2a_{\alpha}}}\right)^m H_m(\mu'_{\alpha}) \end{aligned}$$

Using intermediate basic functions we get for joint probability:

$$\begin{aligned} P(\lambda_{\alpha}, \lambda_{\beta}) &= \int \left( \prod_{i=2}^N d\lambda_i^{(\alpha)} d\lambda_i^{(\beta)} \right) \left( \prod_{i=1}^N \prod_{\gamma=\alpha+1}^{\beta-1} d\lambda_i^{(\gamma)} \right) \times \\ &\quad \times \det_{ij}[\xi_i^{(\alpha)}(\lambda_j^{(\alpha)})] \det_{ij}[\eta_i^{(\beta)}(\lambda_j^{(\beta)})] e^{S-S_{\alpha}-S'_{\beta}} \end{aligned} \quad (3.15)$$

Integrating over intermediate eigenvalues  $d\lambda_i^{(\gamma)}$ ,  $\gamma = \alpha + 1, \dots, \beta - 1$  we obtain the joint probability of two-matrix model for which we already know the result. Hence we get the result (3.2). All derivation above is valid also for more general potentials, polynomial-like  $V_{\alpha}(\tau) = \sum_{k=1}^{p_{\alpha}} t_k \tau^k$  or not. The sufficient ingredients are the coefficients of the  $Q$ -matrices.

## 4 Star-matrix model

We study the star-matrix model with partition function:

$$Z = \int \prod_{i=1}^N (d\lambda_i^{(0)} \prod_{\alpha=1}^q d\lambda_i^{(\alpha)}) \prod_{i < j} [(\lambda_i^{(0)} - \lambda_j^{(0)}) \prod_{\alpha=1}^q (\lambda_i^{(\alpha)} - \lambda_j^{(\alpha)})] \times \\ \times \exp(\sum_{i=1}^N (V_0(\lambda_i^{(0)} + \sum_{\alpha=1}^q V_{\alpha}(\lambda_i^{(\alpha)})) + \sum_{\alpha=1}^q c_{\alpha} \lambda_i^{(0)} \lambda_i^{(\alpha)}) \quad (4.1)$$

We define the orthogonal polynomial basis as  $\xi_n$  and (instead of one conjugate polynomial  $\eta_m$   $q+1$  polynomials  $\eta_m^{(\alpha)}$ :

$$\int d\lambda^{(0)} \prod_{\alpha=1}^q d\lambda^{(\alpha)} \xi_n^q(\lambda^{(0)}) e^{V_0 + \sum_{\alpha=1}^q (V_{\alpha} + c_{\alpha} \lambda^{(0)} \lambda^{(\alpha)})} \prod_{\alpha=1}^q \eta_{m_{\alpha}}^{(\alpha)}(\lambda^{(\alpha)}) = h_n \delta_{nm}, \\ m = m_{\alpha}, \alpha = 1, \dots, q. \quad (4.2)$$

This basis is unusual but it works quite well at least for gaussian potentials:  $V_{\alpha}(\tau) = t_{\alpha} \tau^2 + u_{\alpha} \tau, \alpha = 0, 1, \dots, q$

We introduce  $Q$ -matrices as:

$$\int d\lambda^{(0)} \prod_{\alpha=1}^q d\lambda^{(\alpha)} \xi_n^q(\lambda^{(0)}) \lambda^{(\alpha)} e^{V_0 + \sum_{\alpha=1}^q (V_{\alpha} + c_{\alpha} \lambda^{(0)} \lambda^{(\alpha)})} \prod_{\alpha=1}^q \eta_{m_{\alpha}}^{(\alpha)}(\lambda^{(\alpha)}) = h_n Q_{\alpha, nm}, \\ m = m_{\alpha}, \alpha = 1, \dots, q. \quad (4.3)$$

The coupling conditions are:

$$qP_0 + 2t_0Q_0 + u_0 + \sum_{\alpha=1}^q c_{\alpha}Q_{\alpha} = 0 \\ \overline{P}_{\alpha} + 2t_{\alpha}Q_{\alpha} + u_{\alpha} + c_{\alpha}Q_0 = 0, \alpha = 1, \dots, q \quad (4.4)$$

With the following parametrization of  $Q$ -matrices:

$$Q_0 = I_+ + a_0 I_0 + a_1 \epsilon_- \\ Q_{\alpha} = b_{\alpha}/R_{\alpha} I_+ + d_{\alpha} I_0 + R_{\alpha} \epsilon_-, \alpha = 1, \dots, q \quad (4.5)$$

we arrive at following equations:

$$2t_{\alpha}R_{\alpha} + c_{\alpha}a_1 = 0 \\ 2t_{\alpha}b_{\alpha} + n + c_{\alpha}R_{\alpha} = 0 \\ 2t_{\alpha}d_{\alpha} + u_{\alpha} + c_{\alpha}a_0 = 0 \\ 2t_0 + \sum \frac{c_{\alpha}b_{\alpha}}{R_{\alpha}} = 0 \\ 2t_0a_0 + u_0 + \sum c_{\alpha}d_{\alpha} = 0 \\ 2t_0a_1 + qn + \sum c_{\alpha}R_{\alpha} = 0 \quad (4.6)$$

Solving the coupling conditions we get :

$$\begin{aligned} a_1 &= -\frac{2q}{A}, a_0 = \frac{1}{A}(\sum \frac{c_\alpha u_\alpha}{t_\alpha} - 2u_0) \\ b_\alpha &= -\frac{1}{2t_\alpha^2}(\frac{c_\alpha^2 q}{A} + t_\alpha), R_\alpha = \frac{c_\alpha q}{t_\alpha A} \\ d_\alpha &= \frac{1}{At_\alpha}(c_\alpha u_0 - 2t_0 u_\alpha + u_\alpha \sum \frac{c_\alpha^2}{2t_\alpha} - c_\alpha \sum \frac{c_\alpha u_\alpha}{2t_\alpha}) \end{aligned}$$

where  $A = 4t_0 - \sum c_\alpha^2/t_\alpha$ .

In the same way we get the basic functions for  $Q$ -matrix model we can obtain them for star matrix model:

$$\begin{aligned} \xi_n(\lambda^{(0)}) &= H_n(\lambda'^{(0)}), \\ \eta_m^{(\alpha)}(\lambda^{(\alpha)}) &= R_\alpha^n H_n(\lambda'^{(\alpha)}), \alpha = 0, 1 \dots q \end{aligned} \quad (4.7)$$

with:

$$\lambda'^{(0)} = \frac{\lambda^{(0)} - a_0}{\sqrt{2a_1}}, \lambda'^{(\alpha)} = \frac{\lambda^{(\alpha)} - d_\alpha}{\sqrt{2b_\alpha}} \quad (4.8)$$

Because these basic functions satisfy relation:

$$\eta_m(\lambda^{(0)}) = \int e^{V_\alpha + c_\alpha \lambda^{(0)} \lambda^{(\alpha)}} \eta_m(\lambda^{(\alpha)}) \quad (4.9)$$

we can integrate over Vandermonde determinants:

$$\det_{ij}[\eta_i^{(0)}(\lambda_j^{(0)})] = \int e^{V_\alpha + c_\alpha \lambda^{(0)} \lambda^{(\alpha)}} \det_{ij}[\eta_i^{(\alpha)}(\lambda_j^{(\alpha)})] \quad (4.10)$$

Then we have for the joint probability of two eigenvalues the simple expression:

$$P(\lambda^{(\alpha)}, \lambda^{(\beta)}) \sim P_{Herm}(\lambda'^{(\alpha)}, \lambda'^{(\beta)}), \alpha, \beta = 0, 1 \dots q \quad (4.11)$$

with  $\lambda', \mu'$  given by equation (4.8).

## 5 Generalized Calogero-Sutherland model

The connection with Calogero model permits the calculation of the joint distribution functions for random multimatrix models for other ensembles, different from the hermitean one.

We obtain the Calogero model related to the 2-matrix model .The eigenvalue problem for Calogero model follows from the heat equation satisfied by the Itzykson-Zuber integral.

We introduce the kernel:

$$K(X, Y|t) = \langle X | e^{-tD} | Y \rangle = (2\pi t)^{-N^2/2} \int dU \exp[-\frac{1}{2t} Tr(X - UYU^+)] \quad (5.1)$$

which is related with the Itzykson-Zuber integral  $K(X, Y|t=1) = \exp(-\frac{1}{2t} \text{Tr}(X^2 + Y^2))I(X, Y)$ :

$$I(X, Y) = \int dU \exp[\text{Tr}(XUYU^+)] = \frac{\det_{ij}(e^{x_i y_j})}{(\Delta(X)\Delta(Y))^{\beta/2}} \quad (5.2)$$

The kernel (5.1) satisfies the heat equation [5][6]:

$$(\frac{\partial}{\partial t} + D_X)\tilde{K}(X, Y|t) = \delta(X, Y) \quad (5.3)$$

where  $\tilde{K}(X, Y|t) = (\Delta(X)\Delta(Y))^{\beta/2} K(X, Y|t)$  and the laplacian is:

$$D_X = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}(\frac{\beta}{2} - 1) \sum_{i < j} \frac{1}{(x_i - x_j)^2} \quad (5.4)$$

Solving equation (5.3) gives:

$$\tilde{K}(X, Y|t) = (2\pi t)^{-N^2/2} \sum_{\sigma} \eta_{\sigma} \exp[-\frac{1}{2t} \sum_i (x_{\sigma(i)} - y_i)^2] \quad (5.5)$$

from which follows the expression for the Itzykson-Zuber integral ( $\sigma$  is the permutation).

We introduce the function:

$$\Phi(X|t) = \int \tilde{K}(X, Y|t) \Phi(Y) dY \quad (5.6)$$

that fulfills the heat equation with initial condition  $\Phi(X|t=0) = \Phi(X)$ .

We can search for stationary solutions in the form  $\Phi(X|t) = \sum_n \Phi_n(X) e^{-E_n t}$  where  $\Phi_n(X)$  satisfies the Calogero equation (without potential term):

$$\left( -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{\beta}{2}(\frac{\beta}{2} - 1) \sum_{i < j} \frac{1}{(x_i - x_j)^2} \right) \Phi_n(X) = E_n \Phi_n(X) \quad (5.7)$$

The eigenvalues of matrix  $X$  are chosen such that  $y_1 < y_2 \dots < y_N$ . These eigenvalues  $y_1 \dots y_N$  are mapped by the kernel  $\tilde{K}(X, Y|t)$  into  $x_{\sigma(1)} \dots x_{\sigma(N)}$ .

For  $t \rightarrow 0$ , the kernel  $\tilde{K}(X, Y|t)$  tends to  $\sum_{\sigma} \eta_{\sigma} \delta^{(N)}(x_{\sigma(i)} - y_i)$ . Hence if we consider  $\Psi(X)$  as a particular solution of Calogero model with  $x_1 < x_2 \dots < x_N$ , the function  $\Phi(X|t=0)$  is the general solution for eigenvalues  $x_i$  in arbitrary order, being the linear combination of functions  $\Psi(\sigma X)$  :

$$\Phi(X|t=0) = \sum_{\sigma} \Psi(\sigma X), \quad \Psi(\sigma X) = \eta_{\sigma} \Psi(X)$$

where  $\sigma$  is the permutation of eigenvalues  $x_i$  ;  $\eta_{\sigma} = -1$  for free fermions ( $\beta = 2$  for hermitean matrices) and  $\eta_{\sigma} = +1$  for free bosons ( $\beta \rightarrow 0$  for harmonic oscillator) .

For  $t \rightarrow \infty$  the dominant contribution is given by the vacuum configuration  $\Phi_0(X)$ . The kernel  $\tilde{K}(X, Y|t)$  plays the role of instanton propagator connecting the initial vacuum configuration  $\Psi_0(Y) = (\Delta(Y))^{\beta/2}$  to final vacuum configuration  $\Phi_0(X) = (\Delta(X))^{\beta/2}$ .

For 2-matrix model we can define the Generalized Calogero system:

$$\left( -\frac{1}{2} \left( \sum_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_i \frac{\partial^2}{\partial \mu_i^2} \right) + \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \sum_{i < j} \left( \frac{1}{(\lambda_i - \lambda_j)^2} + \frac{1}{(\mu_i - \mu_j)^2} \right) + \sum_i (V_1(\lambda_i) + V_2(\mu_i) + c\lambda_i\mu_i) \right) \Phi_n(\lambda) \Psi_m(\mu) = E_{nm} \Phi_n(\lambda) \Psi_m(\mu) \quad (5.8)$$

When  $c = 0$  the generalized system splits into two Calogero systems :

$$\begin{aligned} \left( -\frac{1}{2} \sum_i \frac{\partial^2}{\partial \lambda_i^2} + \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \sum_{i < j} \frac{1}{(\lambda_i - \lambda_j)^2} + \sum \lambda'^2 \right) \Phi_n(\lambda) &= E_{1,n} \Phi_n(\lambda) \\ \left( -\frac{1}{2} \sum_i \frac{\partial^2}{\partial \mu_i^2} + \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \sum_{i < j} \frac{1}{(\mu_i - \mu_j)^2} + \sum \mu'^2 \right) \Phi_n(\mu) &= E_{2,m} \Phi_n(\mu) \end{aligned} \quad (5.9)$$

The ground states can be written in terms of the eigenfunctions (2.14):

$$\begin{aligned} \Phi_0(\lambda) &= (\det_{ij} \xi_i(\lambda_j))^{\beta/2} \exp\left(-\sum_i \lambda_i'^2/2\right) \\ \Psi_0(\mu) &= (\det_{ij} \eta_i(\mu_j))^{\beta/2} \exp\left(-\sum_i \mu_i'^2/2\right) \end{aligned} \quad (5.10)$$

We can see that the probability of amplitudes (5.10) is the partition function of the 2-matrix model:

$$Z = \int d\lambda_1 \dots d\lambda_N |\Phi_0(\lambda)|^2 = \int d\mu_1 \dots d\mu_N |\Psi_0(\mu)|^2$$

The system (5.10) permits us to calculate the joint probability  $P(\lambda, \mu)$  for general ensemble. It coincides with formula (2.18) (for  $c = 0$ ) where we replace  $N$  by  $\beta N/2$ :

$$P(\lambda, \mu) = \sqrt{\beta N/2 - \lambda'^2} \sqrt{\beta N/2 - \mu'^2} \quad (5.11)$$

## 6 Conclusions

These models present interest in the study of quantum chaos for  $q$  systems interacting in various ways. The density of levels depends on the total energy which behaves like  $N$ , for large  $N$ . The interaction of  $q$  subsystems redistribute the energy between the subsystems and change in non-trivial way the joint distribution functions. Different kinds of interaction (chain or star- type ) give different probabilities for energy levels.

### ACKNOWLEDGEMENTS

I would like to thank Prof.L.Bonora and Dr.F.Nesti for many usefull discussions. I also want to aknowledge the SISSA staff and students for the stimulating enviroment.

## References

- [1] C.E.Porter,*Statistical Theories of Spectra: Fluctuations,a collection of reprints*,(  
Academic Press, New York and London,1965)
- [2] M.L.Mehta,*Random Matrices*, (Academic Press,New York and London,1967)
- [3] S.Kharchev et al., Nucl.Phys.**B 380**,181 (1992)
- [4] L.Bonora, F.Nesti, E.Vinteler, " *The multimatrix models and matrix chains*",  
preprint SISSA (unpublished)
- [5] M.L.Mehta, Comm.Math.Phys. **79**,327(1981)
- [6] C. Itzykson, J.B.Zuber, J.Math.Phys. **21**, 411 (1980)

### FIGURE CAPTIONS

Figure 1. -Represents the level density  $P(x, x)$  in terms of the energy  $x = \alpha\lambda$  and the asymmetry  $y = N\epsilon$ . For  $y = 0$  we have the semicircular law  $P(x, x) = \sqrt{2N - x^2}$  and for small  $y \neq 0$  we get the oscilations of level density.